# Mathcad interactive didactic tool for studying the trajectory of a satellite around the Earth

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## Abstract

*The manuscript describes an interactive teaching tool made with worksheets in Mathcad for studying the trajectory of a satellite around the Earth. The satellite is modeled as a mass point, while the rotational motion of the Earth around its own axis is neglected. Depending on the initial conditions specified by the launch altitude, the initial velocity and the angle between the initial velocity vector and the initial vector radius, the trajectory of the satellite is plotted. It is shown how the facilities of Mathcad worksheets for numerical calculation and for graphical representation in polar coordinates can be explored. The modification of the initial conditions allows for the observation of the feedback on the elliptical trajectory. Thus, there can also be obtained the particular situations corresponding to the ballistic trajectory or the escape from the terrestrial gravitational field when the trajectory becomes a parabola or a hyperbola. The use of this tool in the classroom helps students to develop their graphic representation skills in Mathcad for the study of the motion in a central force field. Students can clarify how the initial conditions determine the trajectory of a mass point in a central force field with gravitational potential.*

**Keywords**: Mathcad, didactic tool, artificial satellite, central force field, gravitational potential, physics education.

## 1    Introduction

The problem of motion in a central force field is one of the most studied problems of classical mechanics given its importance for astronomy and astrophysics, as well as atomic physics. For the teaching and learning of this topic, a series of didactic tools have been elaborated, thoroughly described in the specialized literature.

The "safety domain" of the elliptical orbits of the satellites, respectively the ballistic trajectories of missiles in the central gravitational field was studied, compared to the parabolic trajectories of the projectiles in the uniform gravitational field. Examples of practical uses of the envelope of the family of orbits were discussed (Butikov, 2015). The n-dimensional problem of Kepler-Coulomb orbits was investigated by generalizing the angular momentum and the Laplace-Runge-Lenz vector in several dimensions. It has been shown that Kepler's three laws remain valid even in higher dimensions by establishing analogies between the physical quantities characteristic of an orbital motion both from a conventional and an n-dimensional approach (Önder and Verçin, 2006). An attempt was made to generalize Bertrand's theorem for the motion in a central force field for relativistic systems. Thus, a stability criterion was deduced for the central potential that produces stable orbits in the relativistic motion. It has been shown that the Coulomb potential passes the stability test for circular orbits at relativistic velocities while the harmonic potential determines the instability of the orbits (Kumar and Bhattacharya, 2011).

The radial motion of an object in the gravitational field generated by a single isotropic mass distribution at the origin was studied. Using the Mathcad program, various star trajectories were analyzed for a given distribution of the galaxy mass (Zürcher and Kaufman, 2011). The PROPAT

Matlab function package to simulate the orbit of a satellite was described and several simulations performed with this package were presented, for example for the 1U CubeSat satellite (Carrara, 2015). In the $C^{++}$ programming environment, using the Qt Framework, the GUI software package was created to simulate the dynamics of satellites near the Earth (Ivanov et al, 2019).

The motion in the central gravitational force field was also approached with Excel spreadsheets. It has been shown that the spreadsheet can provide concrete models for the motion of the satellite. Thus, using the iterative method of solving the equations of motion in the gravitational field, the trajectory of the satellite can be graphically represented in the spreadsheet (Quale, 2012). In the approximation of the constant gravitational field and in the presence of a resistive medium, tools were described both for the study of vertical motion (Grigore and Barna, 2015; Grigore et al, 2017 b) and for the study of projectile motion (Grigore et al, 2017 a).

This paper describes an interactive teaching tool made with worksheets in Mathcad to study the trajectory of a satellite around the Earth. The satellite is considered a point mass while neglecting the rotational motion of the Earth around its own axis. Depending on the input data, the trajectory of the satellite launched into orbit around the Earth can be reproduced. The modification of the initial conditions allows the observation of the feedback on the elliptical trajectory. Also, particular cases can be obtained corresponding to the ballistic trajectory or the escape from the terrestrial gravitational field when the trajectory becomes a parabola or a hyperbola.

## 2    Theoretical background

We consider a body of mass m, modeled as a point mass, which moves in the gravitational field of a massive body of mass M, where M>>m. In relation to an inertial reference system related to the mass body M, the trajectory of the point mass is a conic whose equation, in polar coordinates (r, θ), is transcribed (Aaron, 2002):

$$[1] \quad r_{(\theta)} = \frac{p}{1 + \varepsilon \cos(\theta - \gamma)}$$

In equation (1) p is the semi-latus rectum of the conic, ε is the eccentricity and γ the anomaly angle.

The measures p, ε and γ are determined by the initial conditions of the motion.

We set the initial conditions by the distance from the center of force, $r_0$, the magnitude of the velocity vector, $v_0$, and the angle $\alpha_0$ between the position vector $\vec{r}_0$ and the velocity vector $\vec{v}_0$ as shown in figure 1.a.
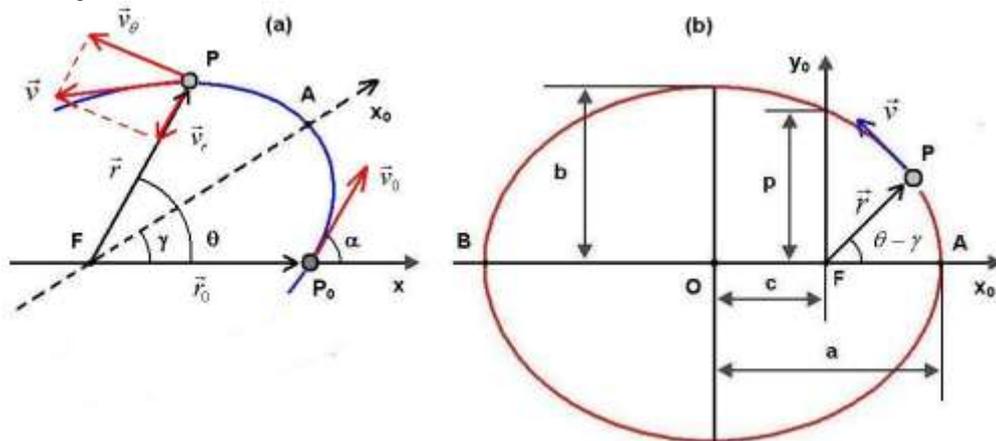
Figure 1. Motion in a central force field: a) Specifying the initial conditions; b) Elliptical trajectory relative to the symmetry axes

The angular momentum of the point mass, L, and the total energy of the system formed by the two bodies, E, are conserved. We have:

$$[2] \quad L = mr^2\dot{\theta} \qquad E = \frac{1}{2}mv^2 - k\frac{mM}{r}$$

where k represents the gravitational constant and v is speed of point mass.

The two constants of motion, L and E, can be transcribed according to the initial conditions:

$$[3] \quad L = mr_0 v_0 \sin\alpha_0 \qquad E = \frac{1}{2}mv_0^2 - k\frac{mM}{r_0}$$

The semi-latus rectum p, the eccentricity $\varepsilon$ and the angle $\gamma$ between the axis Ox and the symmetry axis $Ox_0$ of the conic is calculated as follows:

$$[4] \quad p = \frac{L^2}{km^2 M} \qquad \varepsilon = \sqrt{1 + \frac{2EL^2}{k^2 m^3 M^2}} \qquad \gamma = arctg\frac{p}{(r_0 - p)tg\alpha_0}$$

If E<0 then the trajectory is an ellipse having one of the foci in the force center. The semi-axes of the ellipse, a and b, the focal distance c and the orbital period T are given by the relations:

$$[5] \quad a = \frac{kmM}{2|E|} \qquad b = \frac{L}{\sqrt{2m|E|}} \qquad c = \varepsilon a \qquad T = 2\pi\sqrt{\frac{a^3}{kM}}$$

During the motion on the elliptical trajectory, the minimum and maximum distance of the point mass from the center of force (distance to the perigee, respectively to the apogee), are given by the equations:

$$[6] \quad r_{min} = a(1 - \varepsilon) \qquad r_{max} = a(1 + \varepsilon)$$

The velocity of the point mass according to the vector radius is calculated taking into account the conservation of the total energy given by equation (2). We obtain:

$$[7] \quad v_{(r)} = \sqrt{\frac{2E}{m} + \frac{2kM}{r}}$$

To calculate the maximum velocity – velocity at perigee, respectively the minimum velocity – velocity at apogee, we replace $r_{min}$, respectively $r_{max}$, from relations [6] in equation [7].

## 3    Organization of Mathcad worksheets

The Mathcad document for studying the motion of a satellite around the Earth is divided into three sections: Input Data, Solution, and Results, each section having its own subsections. Figure 2 shows the Mathcad sheet with the Input and Solution Data sections.

Figure 2. Mathcad worksheet with the sections Input Data and Solution

The input data are grouped into two subsections, General data and Initial conditions. In the General data subsection the following parameters are introduced: gravitational constant, k; radius of the Earth, RE; mass of the Earth, M; satellite mass, mo. In the Initial conditions subsection the following parameters are introduced: altitude, h; initial velocity, vo; launch angle αo which represents the angle between the vectors $\vec{v}_0$ and $\vec{r}_0$. The numerical value of each parameter from the input data has been multiplied by the corresponding unit of measure considering that the Mathcad program also includes the units of measurement in the calculations.

In the Solution section we have written the calculation formulas taking into account the order in which the Mathcad program reads them, namely, from top to bottom and from left to right. This feature of the program allows a compact dimensioning of the solution of the approached problem.
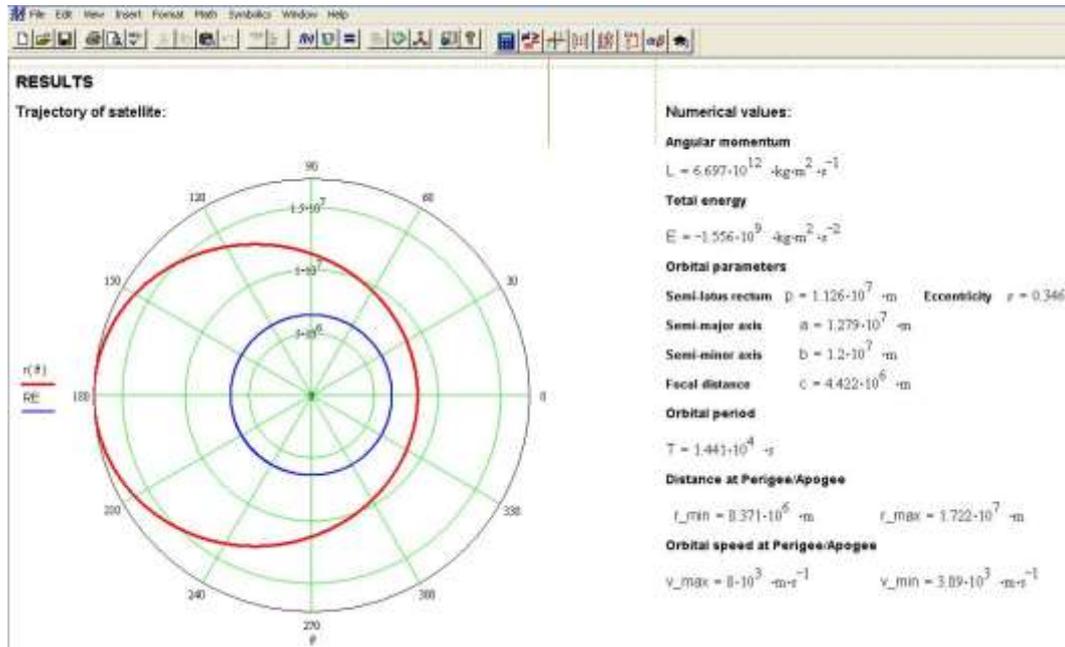
Figure 3. Mathcad worksheet with the Results section. The trajectory of the satellite around the Earth for $m_0$=100 Kg, h=2000 Km, $v_0$=8 Km/s, $\alpha_0$=90°

On the first line of the Solution section we have written the Mathcad formula for the magnitude of the initial vector radius, $r_0$, as the sum between the Earth radius and the launch altitude:

[8] $ro := RE + h$

On the second line we have written the Mathcad formulas to calculate the angular momentum, L, and the total energy, E. To do so, we have transposed in the Mathcad worksheet equations (3) taking into consideration the initial conditions set in the input data:

[9] $L := mo \cdot ro \cdot vo \cdot \sin(\alpha o)$ $\qquad E := \left(\dfrac{1}{2}\right) \cdot mo \cdot vo^2 - k \cdot mo \cdot \dfrac{M}{ro}$

Next, on the third line, we have written the Mathcad formulas to calculate the semi-latus rectum p, eccentricity ε and the anomaly angle γ according to equations (4).

On the fourth line of the Solution section we have written the Mathcad formula to calculate the equation of the trajectory in polar coordinates in conformity with relation (1). On the following lines we wrote the Mathcad formulas for the calculation of the trajectory parameters, of the orbital period and of the distances and velocities at perigee and apogee according to equations (5)-(7).

The Results section comprises two subsections:

- In the first subsection, as seen on the left side of figure 3, the trajectory of the satellite in a reference system associated with the polar coordinates is shown. The trajectory curve is colored red and the circle associated with the surface of the Earth is colored blue. It is observed that the center of the Earth is one of the foci of the ellipse.

- In the second subsection, shown on the right side of figure 3, the numerical values for the following parameters were calculated: the angular momentum L, the total energy E, the semi-latus rectum p; the eccentricity ε, the semi-axes of the ellipse a, b and the focal

distance c; the orbital period T; the minimum distance $r_{min}$ and the maximum distance $r_{max}$ from satellite to center of force; the maximum velocity $v_{max}$ and the minimum velocity $v_{min}$ of the satellite in orbit (the velocity at perigee and the velocity at apogee). The units of measurement of the calculated quantities are rendered in the International System.
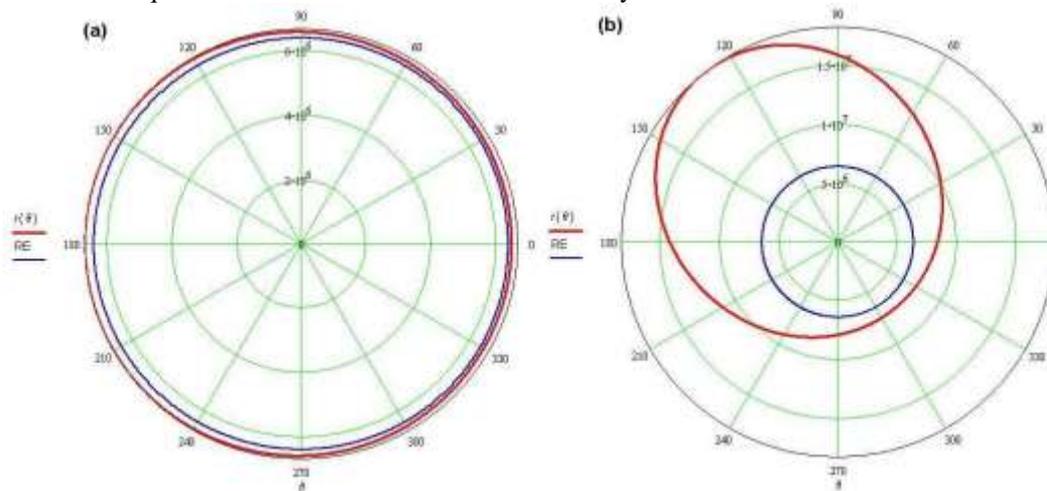


Figure 4. a) Trajectory of the satellite for h=100 Km, $v_0$=7.9 Km/s, $\alpha_0$=90°; b) Trajectory of the satellite for h=2000 Km, $v_0$=8 Km/s, $\alpha_0$=75°

For the construction of the satellite trajectory we have used the Insert menu, chosen the Graph option, and from the drop-down menu we have opted for the Polar Plot. Following the Polar Plot command, in the current position of the cursor on the surface of the Mathcad window, the construction area of the graph in polar coordinates was inserted.

In the place-holder associated with the horizontal axis we entered the angular coordinate θ and in the place-holder associated with the vertical axis we entered the parameters RE, the radius of the Earth, and r(θ) - the function associated with the equation of trajectory in polar coordinates given by equation 1.

With the help of the Mathcad tool we further describe various situations for launching a satellite with a mass $m_0$=100 Kg:

A. Launch from the height h=2000 Km with velocity $v_0$=8 Km/s below the launch angle $\alpha_0$=90°. This is the situation shown in figure 3. In the subsection on the left of the figure is shown the trajectory of the satellite and in the subsection on the right the numerical values for the measures characteristic of the orbital motion.

B. Launch from the height h=100 Km with the velocity $v_0$=7.9 Km/s below the launch angle $\alpha_0$=90°. Basically, we can consider that the satellite is launched close to the surface of the Earth. In this case we obtain the trajectory from figure 4.a. It is observed that the ellipse closely follows the surface of the Earth, i.e. it can be approximated with a circular trajectory. We are close to the limit state of the first cosmic speed.

C. Launch from the height h=2000 Km with the velocity $v_0$=8 Km/s below the launch angle $\alpha_0$=75°. In this case the elliptical trajectory is inclined to the axis from which the angular coordinate θ is measured, as seen in figure 4.b.

D. Launch from h=100 Km with the velocity $v_0$=8 Km/s below the launch angle $\alpha_0$=60°.

This is the case of the ballistic trajectory. The motion of the body is delimited by the launch point and the point where the ellipse intersects the surface of the Earth as seen in figure 5.a.
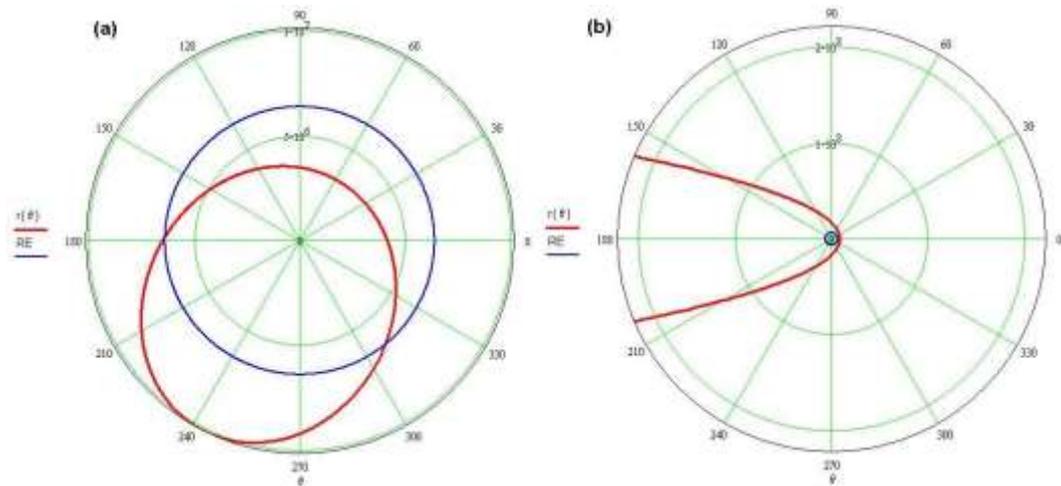


Figure 5. a) Ballistic trajectory for h=100 Km, $v_0$=8 Km/s, $\alpha_0$=60°; b) Satellite parabolic trajectory for h=2000 Km, $v_0$=9.76 Km/s, $\alpha_0$=90°

E.   Launch at height h=2000 Km with the velocity $v_0$=9.76 Km/s under the launch angle $\alpha_0$=90°. In this case, the trajectory of the satellite is parabolic as shown in figure 5.b.

**Conclusions**

The use of this tool in the classroom helps students to develop their graphic representation skills in Mathcad for the study of the motion in the central force field. Students thus explore the polar coordinate system and the modalities of graphical representation in this coordinate system. Students can clarify how the initial conditions determine the trajectory of a point mass in a central force field with gravitational potential.

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Computer programs:  Mathcad 7.0